
IV. *A Letter from Mr.* Colin Mac Laurin, *Profeſſor of Mathematicks at* Edinburgh, *and* F.R.S. *to* Martin Folkes, *Eſq;* V. *Pr. R. S. concerning Æquations with impoſſible Roots.*

SIR,

I Wrote to you laſt Winter, that I had thought of a very eaſie and ſimple way of demonſtrating Sir *Iſaac Newton*'s Rule, by which it may be often diſcover'd when an Æquation has impoſſible Roots. This Method requiring nothing but the common Algebra, and being founded on ſome obvious Properties of Quantities demonſtrated in the following *Lemmata*, without having recourſe to the Conſideration of any Curve whatſoever, which does not ſeem ſo proper a Method in a Matter purely Algebraical, I hope it will not be unacceptable.

Lemma 1. The Sum of the Squares of two real Quantities is always greater than twice their Product. Thus a^2+b^2 is greater than $2ab$; becauſe the Exceſs a^2+b^2-2ab is eqaal to $\overline{a-b}^2$, and therefore is Poſitive; ſince the Square of any real Quantity, Negative or Poſitive, is always Poſitive.

Lemma 2. The Sum of the Squares of three real Quantities is always greater than the Sum of the Products, that can be made by multiplying any two of them. Thus $a^2+b^2+c^2$ is always greater than $ab+ac+bc$; for 'tis plain, that the Exceſs $a^2+b^2+c^2-ab-ac-bc$ =

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$$=\frac{2a^2+2b^2+2c^2-2ab-2ac-2bc}{2}=\frac{a^2-2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2}{2}=\frac{\overline{a-b}^2+\overline{a-c}^2+\overline{b-c}^2}{2}$$ that is, half the Sum of the Squares of the Differences of the Quantities a, b, c: But ſince theſe Squares are Poſitive, it follows, that the Exceſs of $a^2+b^2+c^2$ above $ab+ac+bc$ is Poſitive, and that the Sum of the Squares of three Quantities muſt be greater than the Sum of the Products made by multiplying any two of them.

Lemma 3. The triple Sum of the Squares of four Quantities is greater than the double Sum of the Products, that can be made by multiplying any two of them; for $3a^2+3b^2+3c^2+3d^2-2ab-2ac-2ad-2bc-2bd-2cd=a^2-2ab+b^2+a^2-2ac+c^2+a^2-2ad+d^2+b^2-2bd+d^2+b^2-2bc+c^2+c^2-2cd+d^2=\overline{a-d}^2+\overline{a-c}^2+\overline{a-d}^2+\overline{b-c}^2+\overline{b-d}^2+\overline{c-d}^2$, the Sum of the Squares of the Differences of the four Quantities a, b, c, d. Therefore $3a^2+3b^2+3c^2+3d^2$ is greater than $2ab+2ac+2ad+2bc+2bd+2cd$, the Exceſs being always Poſitive.

Lemma 4. Let the Number of the Quantities a b, c, d, e, &c. be m, the Sum of their Squares A, and the Sum of the Products made by multiplying any two of them B. Then ſhall $\frac{\overline{m-1}\times A}{2}$ be always greater than B.

For by adding together the Squares of the Differences $a-b$, $a-c$, $a-d$, $b-c$, $b-d$, $c-d$, &c. you add a^2 as often to it ſelf as there are Quantities more than a; the ſame is true of b^2, c^2, d^2, e^2, &c. But the Rectangles $-2ab$, $-2ac$, $-2ad$, $-2bc$, $-2bd$, &c. ariſe but once each. Therefore the Sum of all the Squares $\overline{a-b}^2$, $\overline{a-c}^2$, $\overline{b-c}^2$, $\overline{b-d}^2$, &c. $=\overline{m-1}\times a^2+\overline{m-1}\times b^2+\overline{m-1}\times c^2$, &c. $-2ab-2ac-2bc$, &c. $=\overline{m-1}\times A-2B$. But $\overline{a-b}^2$

+

$+\overline{a-c}^2 + \overline{a-d}^2$. &c. is always a Poſitive Quantity; therefore $\overline{m-1}\times A - 2B$ is Poſitive, and conſequently $\frac{\overline{m-1}\times A}{2}$ greater than B.

Cor. It appears from the Demonſtration that the Exceſs of $\overline{m-1}\times A$ above $2B$ is always equal to the Sum of the Squares of the Differences of the Quantities a, b, c, d, &c and that when the Quantities a, b, c, d &c. are all equal, then $\overline{m-1}\times A - 2B = o$, and with this reſtriction the preceding *Lemmata* muſt be underſtood.

It is to be obſerved, that tho' we have ſuppoſed in theſe *Lemmata* the Quantities a, b, c, d, &c. Poſitive, they are *a fortiori* true of Negative Quantities, whoſe Squares are the ſame as if they were Poſitive, while the Sum of their Products is either the ſame, or leſs than it would be, were they all Poſitive.

PROP. I.

In a Quadratic Æquation that has its Roots real, the Square of the ſecond Term muſt be always greater than the quadruple Product of the third and firſt Terms.

Let the Roots of the Quadratic Æquation be repreſented by $+a$ and $+b$; and if x be the unknown Quantity, then ſhall

$$\begin{array}{l} x^2 - ax + ab = o \\ \quad\;\; - bx \end{array}$$

Now ſince $a^2 + b^2$ is greater than $2ab$, by *Lemma* 1, therefore $a^2 + b^2 + 2ab$ is greater than $4ab$; therefore $\overline{a+b}^2 \times x^2$, the Square of the ſecond Term, will be greater than $4ab \times x^2$ the Quadruple Product of the firſt and third Terms.

PROP. II.

In any Cubic Æquation, all whoſe Roots are real, the Square of the ſecond Term is always greater than the triple Product of the firſt and third.

If the Cubic Æquation has all its Roots real they may be repreſented with their Signs by a, b, c, and the Æquation will be expreſſed thus:

$$\begin{array}{l} y^3 - a y^2 + a b y - a b c = 0 \\ - b y^2 + a c y \\ - c y^2 + b c y \end{array}$$

But by *Lemma* 2, $a^2+b^2+c^2$ is always greater than $ab+ac+bc$; and conſequently adding $2ab+2ac+2bc$ to both ſides, $a^2+b^2+c^2+2ab+2ac+2bc$ $(=\overline{a+b+c}^2)$ will be greater than $3ab+3ac+3bc$; and therefore $\overline{a+b+c}^2 \times y^4$ muſt be greater than $\overline{3ab+3ac+3bc} \times y^4$, that is, the Square of the ſecond Term muſt be greater than the triple Product of the firſt and third Terms.

Cor. 1. In general, it appears from the Demonſtration, that the Square of the Sum of three real Quantities, $\overline{a+b+c}^2$ is always greater than the triple Sum of all the Products, that can be made by multiplying any two of them into one another.

Cor. 2. It follows from the Propoſition, that when the Square of the ſecond Term is not greater than the triple Product of the firſt and third Terms, the Roots of the Æquation cannot be all real; but two of them muſt be impoſſible: And this plainly coincides with one Part of Sir *Iſaac Newton*'s Rule for diſcovering when the Roots of Cubic Æquations are impoſſible.

He deſires you may write above the middle Terms of the Æquation the Fractions $\frac{1}{3}$, $\frac{1}{3}$ as in the Margin; and placing the Sign $+$ under the firſt and laſt Term, he multiplies the Square of the ſecond Term by the Fraction $\frac{1}{3}$ that is above it; and if the Product is greater than the Product of the adjacent Terms, he places $+$ under the ſecond Term; but if that Product is leſs, he places $-$ under the ſecond Term, and ſays, there are as many impoſſible Roots as changes in the Signs. Now by this Propoſition, if $p^2 x^4$ is not greater than $3 q x^4$, or $\frac{1}{3} p^2 x^4$ greater than $q x^4$, the Roots cannot be all real. The ſame Suppoſition makes two Changes in the Signs, whatever Sign you place under the third Term, ſince the Signs under the firſt and laſt are both $+$; and therefore this Propoſition demonſtrates the firſt Part of Sir *Iſaac Newton*'s Rule, as far as it relates to Cubic Æquations.

$$\begin{array}{ccccccccc} & & \frac{1}{3} & & \frac{1}{3} & & & & \\ x^3 & + & px^2 & + & qx & + & r & = & 0 \\ + & & - & & * & & + & & \end{array}$$

Cor. 3. If the ſecond Term is wanting in a Cubic Æquation, and the third is Poſitive, two of the Roots of the Æquation muſt be impoſſible: For the Square of the ſecond Term (equal to nothing in this Caſe) will be leſs than the triple Product of the adjacent Terms. But this will better appear from conſidering that, when the ſecond Term vaniſhes in an Æquation, the Poſitive and Negative Roots are equal, and when added together, deſtroy each other. Suppoſe the Roots to be $+a$ and $-b$, $-c$; then in this Caſe $a = +b + c$, and the Coefficient of the third Term will be $-ab - ac + bc = -b^2 - 2bc - c^2 + bc = -b^2 - bc - c^2$, and conſequently Negative. Or, if you ſuppoſe two Roots Poſitive and one Negative, let them be $-a$, $+b$, $+c$, then the Coefficient of the third Term will be ſtill $-b^2 - bc - c^2$. Therefore when the Roots are real, the

the Coefficient of the third Term is negative; and if the Coefficient of the third Term is not affected with a negative Sign, it is a Proof that two of the Roots are Impoſſible.

PROP. III.

In any Cubic Æquation, all whoſe Roots are real, the Square of the third Term muſt be greater than the triple Product of the ſecond and fourth Terms.

In the ſame Cubic Æquation, whoſe Roots are a, b, c, the Square of the third Term is $\overline{ab+ac+bc}^2$, the Product of the ſecond and fourth Terms is $a^2bc+ab^2c+abc^2$, as is plain from the Inſpection of the Æquation; and it is obvious that $a^2bc+ab^2c+abc^2$ is the Sum of the Products of any two of the Terms ab, ac, bc; and therefore by *Corol.* 1 *Prop.* 2. the Square of the Sum of theſe Terms, that is, $\overline{ab+ac+bc}^2$ muſt be greater than $3a^2bc+3ab^2c+3ac^2b$. So that $\overline{ab+ac+bc}^2\times y^2$ muſt be greater than $\overline{3a^2bc+3ab^2c+3ac^2b}\times y^2$; that is, the Square of the third Term muſt be greater than the triple Product of the ſecond and fourth Terms.

Cor. 1. It follows from the Demonſtration, that $\overline{ab+ac+bc}^2$ is always greater than $3abc\times\overline{a+b+c}$.

Cor. 2. If the Square of the third Term is found to be leſs than the triple Product of the ſecond and fourth Terms, then the Roots of the Æquation cannot be all real Quantities; and this concludes with the ſecond Part of Sir *Iſaac Newton*'s Rule for finding when the Roots of a Cubic Æquation are impoſſible: For this Caſe gives — to be placed under the third Term, and con-

$$\begin{array}{l} x^3+\overset{\frac{1}{3}}{p}x^2+\overset{\frac{1}{3}}{q}x+r=0 \\ +\;*\qquad\quad -\quad\; + \end{array}$$

conſequently two Changes of the Signs, whatever Sign is placed under the ſecond Term.

Schol. After the ſame manner, it may be demonſtrated, that in a Cubic Æquation, whoſe Roots are all real, if the ſecond Term is wanting, the Cube of the third Part of the third Term taken poſitively, is always greater than the Square of half the laſt Term. Suppoſe that the Roots of the Æquation are $+a$, $-b$, $-c$, or $-a$, $+b$, $+c$, and that $a=b+c$, then the ſecond Term in the Æquation will be wanting, and the other Terms will be expreſſed thus:

$$\begin{array}{l} y^3 * -b^2 y \pm bc \times \overline{b+c} \\ \quad\;\; -bcy \\ \quad\;\; -c^2 y \end{array}$$

The Square of $b-c$ is always poſitive, ſince b and c are real Quantities. Suppoſe it, (*viz.* $b^2-2bc+c^2$) equal to D, then $b^2+bc+c^2=D+3bc$, and $\overline{b+c}^2=D+4bc$. Therefore $\frac{\overline{b^2+bc+c^2}^3}{27}=\frac{D^3}{27}+\frac{D^2bc}{3}+Db^2c^2+b^3c^3$, and $b^2c^2\times\frac{\overline{b+c}^2}{4}=\frac{Db^2c^2}{4}+b^3c^3$. Now 'tis obvious that $\frac{D^3}{27}+\frac{D^2bc}{3}+Db^2c^2+b^3c^3$ is greater than $\frac{Db^2c^2}{4}+b^3c^3$, ſince D is poſitive, and bc alſo poſitive, b and c being Roots having the ſame Sign. Therefore the Cube of $\frac{1}{3}$ of the third Term having its Sign changed $\left(=\frac{\overline{b^2+bc+c^2}^3}{27}\right)$ is always greater than the Square of half the laſt Term $\left(=b^2c^2\times\frac{\overline{b+c}^2}{4}\right)$. In the Cubic Æquation $x^3*+qx+r=0$, if q be poſitive, or if it

be

be negative and $\frac{+q^3}{27}$ be leſs than $\frac{1}{4}r^2$, it appears that two Roots of the Æquation muſt be impoſſible, from this Corollary, and from *Cor.* 3. *Prop.* 2. taken together.

P R O P. IV.

In a Biquadratic Æquation, all whoſe Roots are real Quantities, $\frac{3}{8}$ of the Square of the ſecond Term is always greater than the Product of the firſt and third Terms; and $\frac{3}{8}$ of the Square of the fourth Term is always greater than the Product of the third and fifth Terms.

1. Let the Æquation be $x^4 - p\,x^4 + q\,x^2 - r\,x + s = 0$; and ſince the Roots are ſuppoſed to be all real, let them be repreſented by a. b, c, d, then $p = a + b + c + d$, and $q = ab + ac + ad + bc + bd + cd$ But it is plain from *Lemma* 3, that $3a^2 + 3b^2 + 3c^2 + 3d^2$ is greater than $2ab + 2ac + 2ad + 2bc + 2bd + 2cd$; and conſequently by adding $6ab + 6ac + 6ad + 6bc + 6bd + 6cd$ to both, we ſhall find that $3 \times \overline{a + b + c + d}^2$ muſt be greater than $8ab + 8ac + 8ad + 8bc + 8bd + 8cd$; that is, $3p^2$ greater than $8q$; and therefore $\frac{3}{8}p^2x^6$ greater than qx^6.

2. Since $r = abc + abd + acd + bcd$, and $s = abcd$; and ſince qs is equal to $a^2b^2cd + a^2c^2bd + a^2d^2bc + b^2c^2ad + b^2d^2ac + c^2d^2ab$, which are the Products can be made of any two of the Quantities abc, abd, acd, bcd, whoſe Sum is r multiplied by one another; it follows, that $3r^2$ is always greater than $8qs$: So that $\frac{3}{8}$ of either the Square of the ſecond Term, or of the Square of the fourth Term, muſt al-

ways be greater than the Product of the Terms adjacent to them.

Cor. Multiply either the Square of the ſecond Term, or the Square of the fourth Term of a Biquadratic Æquation by $\frac{3}{8}$, and if the Product does not exceed the Product of the adjacent Terms, ſome of the Roots of that Æquation muſt be impoſſible.

PROP. V.

In an Æquation of any Dimenſion expreſſed by m, the Coefficients of the ſecond, third, laſt, laſt but one, and laſt but two Terms, being reſpectively A, B, E, D, C if the Roots of the Æquation are all real, then ſhall $\overline{m-1}\times A^2$ always be greater than $2mB$, and $\overline{m-1}\times D^2$ greater than $2mCE$.

1. For ſuppoſing the Roots to be a, b, c, d, e, &c. then by *Lemma* 4, ſhall $\overline{m-1}\times a^2+\overline{m-1}\times b^2+\overline{m-1}\times c^2$ &c. be greater than $2ab+2ac+2ad$, &c. and adding $\overline{2m-2}\times ab+\overline{2m-2}\times ac+\overline{2m-2}\times ad$, &c. to both, the Sum $\overline{m-1}\times a^2+\overline{2m-2}\times ab+\overline{m-1}\times b^2+$ &c. ($=\overline{m-1}\times\overline{a+b+c,\ \&c.}^2$) muſt be greater than $2mab+2mac+2mad$, &c. that is, $\overline{m-1}\times A^2$ muſt be greater than $2mB$.

2. In general, it follows from this Demonſtration, that the Square of the Sum of any Quantities whoſe Number is (m) multiplied by $m-1$, muſt be greater than the Sum of all the Products can be made by multiplying any two of them, multiplied by $2m$. But it is eaſie to ſee from the Geneſis of Æquations, that CE is the Sum of the Products can be made by multiplying any two of the Terms whoſe Sum is D: From which it follows, that $\overline{m-1}\times D^2$ muſt be always greater than $2mCE$.

To be continued.